OPTIMAL QUANTIZATION FUNCTIONS FOR MULTI-LEVEL DIGITAL CORRELATORS

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Abstract. The designs for the new ALMA, EVLA, Arecibo, and GBT digital correlators incorporate multi-level quantization schemes (including 8-level, 9-level, and 16-level modes of operation) for which no complete analyses exist in the radio science literature. Here I derive explicit formulas for the expected output, the variance, and the efficiency (in the weak correlation limit) of an n-level by n-level digital correlator, for arbitrary n, assuming bivariate normal inputs. I also describe Mathematica code I have written that can be used for the derivation and manipulation of such formulas, as well as for the computation, by numerical optimization, of optimal quantization functions. I present explicit numerical results for the cases $n = 2, 3, 4, \ldots, 32$; and $n = 64$, 128, 256, 512, and 1024. I show that analogous, and precise, results for the hybrid cases, m-level by n-level, follow easily from the $m \times m$ and $n \times n$ results.

I. Introduction

The designs for the new ALMA, EVLA, Arecibo, and GBT digital correlators incorporate multi-level quantization schemes. The GBT and Arecibo correlators can operate in 3-level by 3-level, 9-level by 9-level, and 3-level by 9-level modes. A design which has been proposed for the EVLA would incorporate 8-bit (256-level) quantization, followed by digital FIR filtering and re-quantization to 8 or 16 levels prior to correlation. The ALMA correlator is likely to operate in 2-bit and 4-bit (4- and 16-level) modes.

With the exception of a 1968 paper by Cole, a 1973 paper by Hagen and Farley [2], and a 1998 paper by Jenet and Anderson [3], most of the existing publications in the radio science research literature on the topic of quantization and digital correlation [3–12] treat only individual cases corresponding to small numbers of quantization levels ($n \leq 4$) or the digital correlation few-level by analog cases (e.g., [9], also [1]). Most of this literature is reviewed in Ch. 8 of Thompson, Moran, and Swenson [13]. Thompson, in one of the ALMA Memoranda [14], derives approximate, but not precise results for correlator efficiency in the cases $n \geq 8$.

Cole's and Hagen and Farley's papers set up the basic framework which is required for a thorough analysis of the problem at hand. Cole's includes approximate results for the two-bit (four level) and three-bit (eight level) cases. Hagen and Farley's paper treats a total of ten cases: one-bit by analog, three-level by analog, two-bit by analog, one-bit by one-bit, one-bit by two-bit, one-bit by three-level, two-bit by two-bit, a modified two-bit scheme, two-bit by three-level, and three-level by three-level. These authors derive the formulas for the expectation of the correlator output, the variance, the correlator efficiency (relative to that of an ideal analog correlator), and the efficiency gain if the data are sampled at a higher rate than the Nyquist. Their methods of approach are general, and easily adaptable to other cases.

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The very recent paper by Jenet and Anderson deals with the particular case of digitization to $n = 2^k$ levels and includes computed numerical results for $k = 2, 3, 4, \ldots, 8$ (i.e., up to 256 levels). Their criterion for selecting an optimal quantization function is that of minimizing the root-mean-square voltage or power distortion of the quantized signal, rather than minimizing the variance of the estimated correlation coefficient of the digitized signals, in the weak-signal limit.

All of the results which I shall present are derived following the same methods used by Cole, Hagen and Farley, and Jenet and Anderson.

II. Preliminaries

Consider the case of two real-valued time series, $X = \{x_1, x_2, \ldots, x_N\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, and suppose the (x_i, y_i) to be drawn from a bivariate normal distribution with correlation coefficient ρ. The sample correlation coefficient, $\hat{\rho}$ is given by

$$\hat{\rho} = \frac{1}{N} \sum_{i=1}^{N} x_i y_i \,. \tag{2-1}$$

[Jon — I haven't written the rest of this section yet. It'll give expressions for the expected correlator output, its derivative, etc. in general form; i.e., for arbitrary n and arbitrary input thresholds and output levels.]

III. Key Results

The results given in this section pertain only to the case of real-valued, stationary, zero-mean, unit-variance, bandlimited, jointly Gaussian inputs to the correlator, with correlation coefficient ρ, sampled at the Nyquist rate. (The assumption of unit variance is merely a convenience, so that σ_x's and σ_y's do not have to divide the values of input thresholds in the formulas below.) The word "efficiency" refers only to the efficiency, relative to an analog correlator, in the weak correlation limit $|\rho| \to 0$. Optimality results are only with regard to correlators employing scalar quantization (as opposed to vector quantization).

Theorem 1. *For even $n \geq 2$, the efficiency of an n-level $\times$ n-level digital correlator with identical quantization functions possessing odd symmetry, whose input thresholds and output levels are*

$$\begin{aligned} Q = \big\{ &\{-v_{n/2-1}, \ldots, -v_1, v_0 (\equiv 0), v_1, \ldots, v_{n/2-1}\}, \\ &\{-w_{n/2}, \ldots, -w_1, w_1, \ldots, w_{n/2}\} \big\}, \end{aligned} \tag{1}$$

with $v_k < v_{k+1}$, is given by

$$\eta = \frac{2 \left(\sum_{i=1}^{n/2} (w_i - w_{i-1}) e^{-v_{i-1}^2/2} \right)^2}{\pi \sum_{i=1}^{n/2} \left(w_i^2 - w_{i-1}^2 \right) \operatorname{erfc}\left(v_{i-1}/\sqrt{2} \right)} \,, \tag{2}$$

where $w_0 \equiv 0$, $v_{n/2} \equiv \infty$, and $\operatorname{erfc} z \equiv 1 - \operatorname{erf} z$ is the complementary error function.

Proof. $\square$

Theorem 2. *For odd $n \geq 3$, the efficiency of an n-level $\times$ n-level digital correlator with identical quantization functions possessing odd symmetry, whose input thresholds and output levels are*

$$\begin{aligned} Q = \big\{ &\{-v_{(n-1)/2}, \ldots, -v_1, v_1, \ldots, v_{(n-1)/2}\}, \\ &\{-w_{(n-1)/2}, \ldots, -w_1, w_0 (\equiv 0), w_1, \ldots, w_{(n-1)/2}\} \big\}, \end{aligned} \tag{3}$$

with $0 < v_1$ and $v_{k-1} < v_k$, is given by

$$\eta = \frac{2 \left(\sum_{i=1}^{(n-1)/2} (w_i - w_{i-1}) e^{-v_i^2/2} \right)^2}{\pi \sum_{i=1}^{(n-1)/2} \left(w_i^2 - w_{i-1}^2 \right) \operatorname{erfc}\left(v_i/\sqrt{2} \right)} \,, \tag{4}$$

where $v_0 \equiv 0$.

Proof. $\square$

Theorem 3. *For even $n \geq 4$, the efficiency of an n-level $\times$ n-level digital correlator, employing identical quantization functions as in Theorem 1, is maximized—for a given set of input thresholds—if the output levels of the quantization function are chosen according to*

$$w_i = \frac{a\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)}\, w_1\,, \qquad i = 2, \ldots, n/2\,, \tag{5}$$

where $w_1 > 0$ may be chosen arbitrarily,

$$a \equiv \frac{e^{v_1^2/2}\operatorname{erf}\left(v_1/\sqrt{2}\right)}{\left(e^{v_1^2/2} - 1\right)}\,, \tag{6}$$

and $v_{n/2} \equiv \infty$.

Proof. Re-write Equation 2 in the form

$$\eta = \frac{2\left(\sum_{i=1}^{n/2} w_i\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right)\right)^2}{\pi\sum_{i=1}^{n/2} w_i^2\left(\operatorname{erfc}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erfc}\left(v_i/\sqrt{2}\right)\right)} \equiv \frac{U}{V}\,. \tag{7}$$

Then substitute the w_i of Equation 5 into the expressions for U and V, and simplify as follows:

$$U = 2\left(\sum_{i=1}^{n/2} w_i\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right)\right)^2 \tag{8a}$$

$$= 2a^2w_1^2\left(\sum_{i=1}^{n/2}\frac{\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right)}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)}\right)^2 \tag{8b}$$

$$= 2a^2w_1^2\left(\sum_{i=1}^{n/2}\frac{\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)^2}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)}\right)^2\,, \tag{8c}$$

and

$$V = \pi\sum_{i=1}^{n/2} w_i^2\left(\operatorname{erfc}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erfc}\left(v_i/\sqrt{2}\right)\right) \tag{9a}$$

$$= \pi a^2w_1^2\sum_{i=1}^{n/2}\frac{\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)^2\left(\operatorname{erfc}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erfc}\left(v_i/\sqrt{2}\right)\right)}{\left(\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)\right)^2} \tag{9b}$$

$$= -\pi a^2w_1^2\sum_{i=1}^{n/2}\frac{\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)^2}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)}\,. \tag{9c}$$

Also, for $2 \leq k \leq n/2$, calculate that

$$\frac{\partial U}{\partial w_k} = 4\left(e^{-v_{k-1}^2/2} - e^{-v_k^2/2}\right)\sum_{i=1}^{n/2} w_i\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right) \tag{10a}$$

$$= 4aw_1\left(e^{-v_{k-1}^2/2} - e^{-v_k^2/2}\right)\sum_{i=1}^{n/2}\frac{\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right)\left(e^{-v_i^2/2} - e^{-v_{i-1}^2/2}\right)}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)} \tag{10b}$$

$$= -4aw_1\left(e^{-v_{k-1}^2/2} - e^{-v_k^2/2}\right)\sum_{i=1}^{n/2}\frac{\left(e^{-v_{i-1}^2/2} - e^{-v_i^2/2}\right)^2}{\operatorname{erf}\left(v_{i-1}/\sqrt{2}\right) - \operatorname{erf}\left(v_i/\sqrt{2}\right)}\,, \tag{10c}$$

and

$$\frac{\partial V}{\partial w_k} = 2\pi w_k \big(\mathrm{erfc}(v_{k-1}/\sqrt{2}) - \mathrm{erfc}(v_k/\sqrt{2})\big)\,, \tag{11a}$$

$$= 2\pi a w_1 \big(e^{-v_{k-1}^2/2} - e^{-v_k^2/2}\big)\,. \tag{11b}$$

Thus $V\frac{\partial U}{\partial w_k} - U\frac{\partial V}{\partial w_k} \equiv 0$ when the w_k are chosen according to Equation 5. So $\frac{\partial \eta}{\partial w_k} = 0$ for all k, which is a necessary condition for maximizing the efficiency η.

Now observe that the expression for η in Equation 7 is a quotient of quadratic forms in the variables w_i. Indeed, the numerator U is the square of a first-order homogeneous expression in the w_i, so it is second-order homogeneous in pairwise products of the w_i, is always non-negative—and is, therefore, a positive semi-definite quadratic form. The denominator V is a sum of positively weighted squares of the w_i (positively weighted because each $v_i > v_{i-1}$) and is, therefore, a positive definite quadratic form. U can be written in the form $U = \mathbf{w}^T A \mathbf{w}$, where A is a symmetric, positive semi-definite matrix. Similarly, V can be written in the form $V = \mathbf{w}^T B \mathbf{w}$, with B symmetric and positive definite. Observe that A is rank deficient—indeed, of rank 1—because $\mathbf{w}^T A \mathbf{w}$ is the square of a first-order homogeneous polynomial in the w_i.

The standard theory of so-called Rayleigh quotients (ratios of quadratic forms) guarantees, because A is of rank 1, that the obviously positive solution given by Equation 5 is a unique maximizer of Equation 7 (up to a constant scaling factor). All other critical points of $\eta = \dfrac{\mathbf{w}^T A \mathbf{w}}{\mathbf{w}^T B \mathbf{w}}$ are eigenvectors corresponding to zero-valued eigenvalues of the generalized eigenvalue problem $A\mathbf{w} = \lambda B \mathbf{w}$, for which the efficiency η is equal to zero (see, e.g., [?], or any comprehensive textbook on advanced linear algebra; also see Theorem 5 below). □

Theorem 4. *For odd $n \geq 5$, the efficiency of an n-level $\times$ n-level digital correlator, employing identical quantization functions as in Theorem 2, is maximized—for a given set of input thresholds—if the output levels of the quantization function are chosen according to*

$$w_k = \frac{a\big(e^{-v_{k+1}^2/2} - e^{-v_k^2/2}\big)}{\mathrm{erf}\big(v_k/\sqrt{2}\big) - \mathrm{erf}\big(v_{k+1}/\sqrt{2}\big)}\, w_1\,, \qquad k = 2, \ldots, (n-1)/2\,, \tag{12}$$

where $w_1 > 0$ may be chosen arbitrarily,

$$a \equiv \frac{\mathrm{erf}\big(v_1/\sqrt{2}\big) - \mathrm{erf}\big(v_2/\sqrt{2}\big)}{e^{-v_2^2/2} - e^{-v_1^2/2}}\,, \tag{13}$$

and $w_{(n-1)/2}$ is evaluated in the limit $v_{(n+1)/2} \to \infty$.

Proof. This theorem is easily proven by following the same sequence of steps as in the proof of Theorem 3. □

Theorem 5. *Consider an n-level $\times$ n-level digital correlator (with $n \geq 4$), employing identical quantization functions—as in Theorem 1 if n is even, or as in Theorem 2 if n is odd. Let A and B be the matrices of the two quadratic forms occurring in the proof of Theorem 3 or Theorem 4, as appropriate. Consider the generalized eigenvalue problem*

$$A\mathbf{z} = \lambda B \mathbf{z}\,. \tag{14}$$

The generalized eigenvectors $\mathbf{z}_i$ are the extremizers of the Rayleigh quotient

$$\eta = \frac{\mathbf{z}^T A \mathbf{z}}{\mathbf{z}^T B \mathbf{z}}\,. \tag{15}$$

Since A is real, symmetric, and of rank 1, and since B is positive definite, the system (14) has one positive real eigenvalue, λ_1, and the rest of the eigenvalues are equal to zero. The *correlator efficiency η is extremized if the output levels of the quantization function are chosen to be proportional to an eigenvector $\mathbf{z}_i$: If $\mathbf{w} = \mathbf{z}_1$ (the eigenvector corresponding to λ_1) the correlator efficiency is λ_1; if $\mathbf{w} = \mathbf{z}_k$, $k \neq 1$, then the correlator efficiency is zero.*

Proof. $\square$

Numerical solutions of the generalized eigenvalue problem (Eq. 14) can be computed via a combination of two standard methods in numerical linear algebra—Cholesky decomposition[1] and Schur decomposition[2], viz.:

Theorem 6. *Let the Cholesky decomposition of the positive definite matrix B be given by $B = R^T R$, and suppose that A is real, symmetric, and positive semi-definite. Set $C = (R^{-1})^T A R^{-1}$, and let the Schur decomposition of C be given by $C = QDQ^T$. Then the columns of $R^{-1}Q$ are the generalized eigenvectors of $A\mathbf{z} = \lambda B\mathbf{z}$ and the diagonal elements of D are the eigenvalues. The eigenvectors extremize the Rayleigh quotient $\dfrac{\mathbf{z}^T A\mathbf{z}}{\mathbf{z}^T B\mathbf{z}}$.*

Proof. $\square$

The previous two theorems are chiefly of pedagogic interest, since the zero-efficiency solutions are of little practical use and Theorems 3 and 4 already provide an analytic solution for the optimal efficiency case. (It would probably be possible to derive analogous analytic solutions for the zero-efficiency eigenfunctions, but—for obvious reasons—I have not felt it worthwhile to do so.) A numerical application of Theorems 5 and 6 is illustrated by:

Example 1. *(See Figure 1.) Consider the case of a 6-level × 6-level correlator. Theorem 4, combined with numerical optimization, can be used to determine the optimal input thresholds ($v_1 = 0.658911$ and $v_2 = 1.44685$) and the optimal output levels ($w_1 = 1$, $w_2 = 3.147795$, and $w_3 = 5.960017$), yielding the optimal efficiency of 94.2022%. The matrices of the two quadratic forms, corresponding to $n = 6$ and the above numerical values of v_1 and v_2, are $A = \begin{pmatrix} 0.076156 & 0.177092 & 0.137024 \\ 0.177092 & 0.411806 & 0.318633 \\ 0.137024 & 0.318633 & 0.246541 \end{pmatrix}$ and $B = \begin{pmatrix} 1.53953 & 0 & 0 \\ 0 & 1.1373 & 0 \\ 0 & 0 & 0.464764 \end{pmatrix}$. Since B is diagonal, its Cholesky factor R is trivial to compute, $R = \begin{pmatrix} 1.24078 & 0 & 0 \\ 0 & 1.06644 & 0 \\ 0 & 0 & 0.681736 \end{pmatrix}$. Then $C = \begin{pmatrix} 0.049467 & 0.133834 & 0.161989 \\ 0.133834 & 0.362091 & 0.438265 \\ 0.161989 & 0.438265 & 0.530464 \end{pmatrix}$, whose Schur decomposition is given by $Q = \begin{pmatrix} 0.229154 & 0.861282 & 0.453522 \\ 0.619981 & 0.230043 & -0.750136 \\ 0.750408 & -0.453071 & 0.481263 \end{pmatrix}$ and $D = \begin{pmatrix} 0.942022 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The single non-zero eigenvalue, D_{11}, is equal to the optimal value of the correlator efficiency. The columns of $R^{-1}Q = \begin{pmatrix} 0.184686 & 0.694147 & 0.365514 \\ 0.581354 & 0.215710 & -0.703400 \\ 1.100730 & -0.664585 & 0.705938 \end{pmatrix}$ are the eigenvectors of $Az = \lambda Bz$, which extremize the Rayleigh quotient. The output levels of the three quantization functions shown in Figure 1 have been chosen accordingly.*

[1] A positive definite matrix A can be factored uniquely in the form $A = R^T R$, where R is an upper triangular matrix with positive diagonal elements (or, equivalently, in the form $A = LL^T$ where L is lower triangular). This is the so-called *Cholesky decomposition* of A. [LINPACK Guide, p. 8.1; Golub and van Loan 1996, p. 143]

[2] We require only the symmetric form of the Schur decomposition theorem: If A is a real, symmetric matrix, then there exists a real orthogonal Q such that $A = QDQ^T$, with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where the λ_i are the eigenvalues of A and the columns of Q the eigenvectors. [Golub and van Loan 1996, p. 393] The Schur decomposition can be computed numerically via the symmetric QR algorithm [Golub and van Loan 1996, pp. 414ff.]

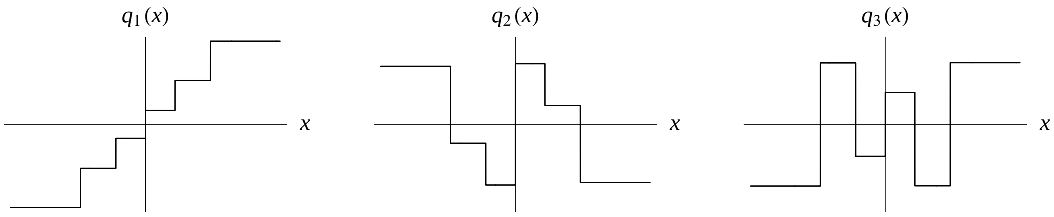

Figure 1. This figure shows the best possible, and two worst possible, 6-level quantization functions. A 6-level $\times$ 6-level digital correlator employing the quantization function q_1 has an efficiency of $\sim$94.20%, which is the best possible. If the quantization function used is q_2 or q_3, the correlator efficiency is exactly zero. These functions were computed according to Theorems 5 and 6 (see Example 1). Any 6-level quantization function that has the same input thresholds as shown here, and which achieves only zero efficiency, is a linear combination of q_2 and q_3.

The following theorem is not new. An equivalent result, with derivation, is given by Bowers and Klingler (1974; their Eq. 10, and surrounding paragraph). None of the other previous literature of which I am aware includes this result:

Theorem 7. *The efficiency η_{Q_1,Q_2} of a digital correlator employing non-identical quantization functions, $Q_1 \neq Q_2$, each one possessing odd symmetry and each one of the form of Equation 1 or Equation 3, is equal to the geometric mean of the efficiencies of a correlator with identical quantization functions $Q_x = Q_y = Q_1$ and a correlator with identical quantization functions $Q_x = Q_y = Q_2$; i.e.,*

$$\eta_{Q_1,Q_2} = \sqrt{\eta_{Q_1,Q_1}\eta_{Q_2,Q_2}}\,. \tag{16}$$

Proof. The right-hand-sides of Equations ? and ? above, the quotient of which is equal to the correlator efficiency, are each separable products of terms in the input levels and output levels of Q_1, and those of Q_2, so this result follows trivially. $\square$

Theorem 7 is of particular interest because the new GBT and Arecibo correlators can operate in 3×3-level, 9×9-level, and 3×9-level modes; and, similarly, the ALMA correlator is likely to operate in 4×4-, 8×8-, and perhaps 4×8-level modes. Bowers and Klingler (1974) describe a correlator that was built in the 1970s and which incorporated 3-level $\times$ 5-level quantization; their paper, however, does not include a formula for, or a plot of, the expected correlator output. My Figure 4 shows the normalized expected correlator output $r(\rho)$ vs. ρ for each of the above cases.

It is interesting to note, from Figure 4, that the inflection of the curve $r(\rho)$ vs. ρ near $\rho = 1$, for the m-level $\times$ n-level case, in some cases is the same as that of the $m\times m$ and $n\times n$ cases—and sometimes different; indeed, the slope at the endpoint may be either ∞ or 0. [I should plot the slope itself for these cases, to make this point.] Note that the curves for the mixed-level cases are "more linear" and that they lie above those corresponding to the $m\times m$ and $n\times n$ cases. And note that for the mixed-level cases the efficiency is smaller than the value of the slope at $\rho = 0$.

IV. Numerical Results

V. Mathematica Software

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Table 1. Optimal Efficiency (in Percent) of an n-Level $\times$ n-Level Digital Correlator, for $n = 2, 3, 4, \ldots, 32; 64, 128, 256, 512, 1024$

n	η_1	η_2	η_3
2	63.6620	—	—
3	80.9826	—	—
4	88.2518	—	88.1154
5	92.0059	91.9641	91.7822
6	94.2022	94.1215	93.9343
7	95.6000	95.4914	95.3140
8	96.5452	96.4189	96.2560
9	97.2147	97.0781	96.9304
10	97.7063	97.5646	97.4313
11	98.0780	97.9348	97.8144
12	98.3660	98.2235	98.1147
13	98.5937	98.4535	98.3547
14	98.7768	98.6398	98.5500
15	98.9263	98.7931	98.7111
16	99.0499	98.9208	98.8457
17	99.1533	99.0285	98.9595
18	99.2407	99.1202	99.0566
19	99.3152	99.1990	99.1402
20	99.3792	99.2673	99.2127
21	99.4347	99.3268	99.2761
22	99.4830	99.3790	99.3318
23	99.5254	99.4252	99.3811
24	99.5628	99.4662	99.4249
25	99.5959	99.5028	99.4640
26	99.6254	99.5356	99.4992
27	99.6517	99.5651	99.5308
28	99.6754	99.5918	99.5594
29	99.6968	99.6160	99.5854
30	99.7161	99.6380	99.6091
31	99.7336	99.6581	99.6307
32	99.7495	99.6765	99.6505
64	99.9354	99.9039	99.8960
128	99.9824	99.9718	99.9696
256	99.9951	99.9918	99.9912
512	—	99.9977	99.9975
1024	—	99.9993	99.9993

Note. η_1 is the optimal efficiency, assuming no constraints. (I have not computed η_1 for the $n = 512$ and $n = 1024$ cases, because of limited computing power.) η_2 is the optimal efficiency if the input thresholds are constrained to be equi-spaced. η_3 is the optimal efficiency if both the input thresholds and the output levels are constrained to be equi-spaced. (In the latter case, the input-threshold constraint is redundant, because the achievement of optimal efficiency under the second constraint implies the first constraint (proof omitted).)

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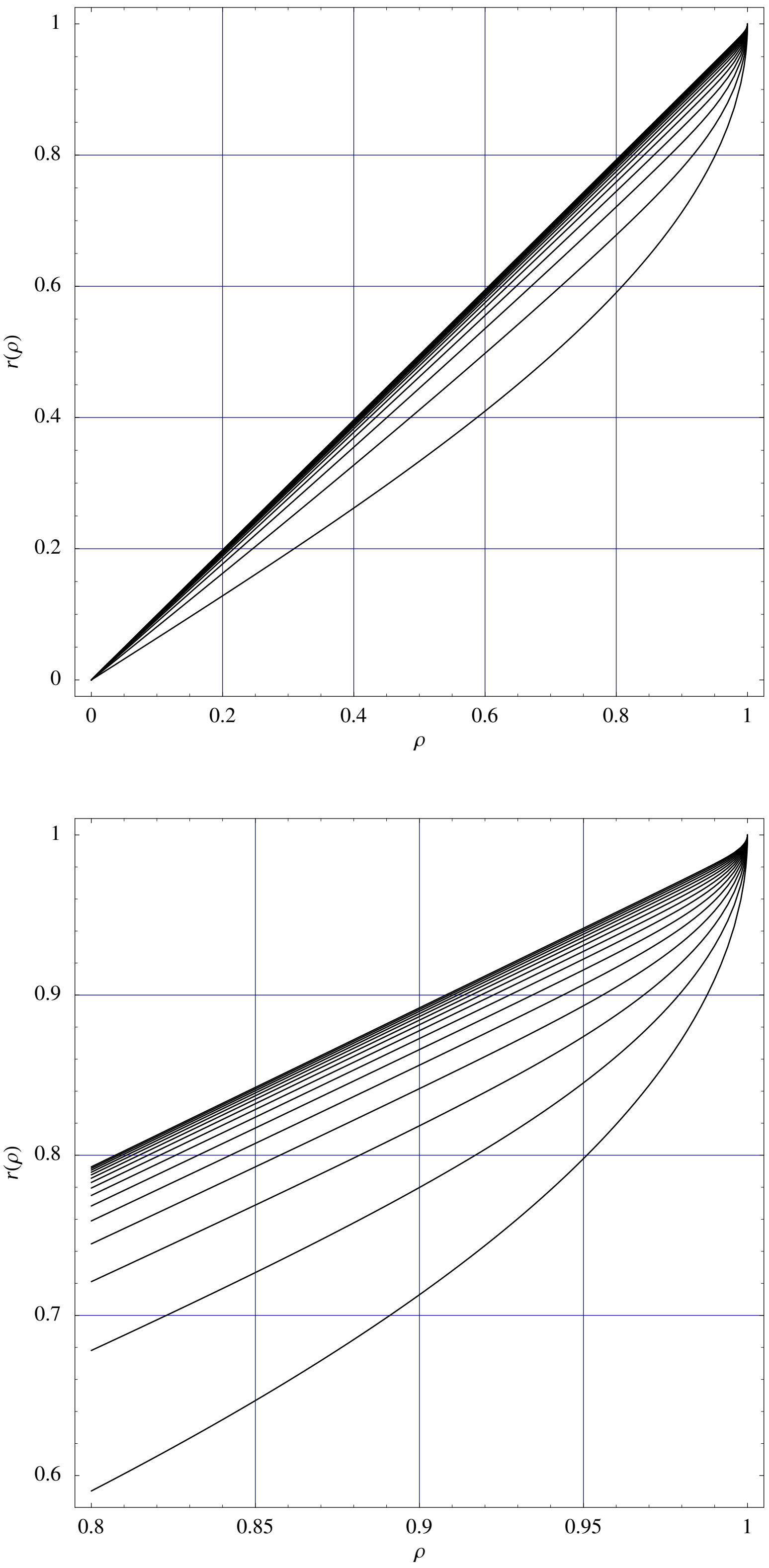

Figure 2. (Top) A plot of the normalized correlator output, $r(\rho)$ vs. ρ, of an n-level $\times$ n-level digital correlator employing the optimal n-level quantization function, computed for the cases *(bottom to top)* $n = 2, 3, 4, \ldots, 16$. **(Bottom)** A blow-up of the region of the top figure corresponding to $0.8 \leq \rho \leq 1.0$.

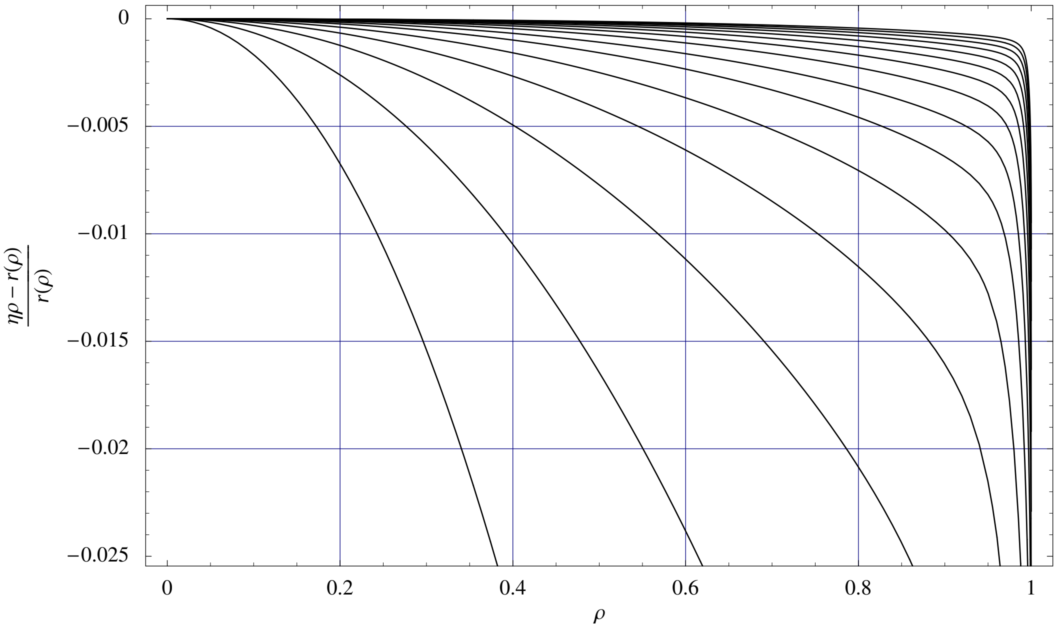

Figure 3. This figure illustrates the departure from linearity of the curves $r(\rho)$ vs. ρ of Figure 2, for $n = 2, 3, 4, \ldots, 16$ *(left to right)*. Specifically, the curves shown here represent the relative error in approximating $r(\rho)$ by a straight line whose slope is equal to $\frac{dr}{d\rho}\Big|_{\rho=0}$. (For an n-level $\times$ n-level digital correlator employing identical quantization functions, this slope is equal to the correlator efficiency η.) Only those relative errors which are smaller than 2.5% (in absolute value) are plotted here.

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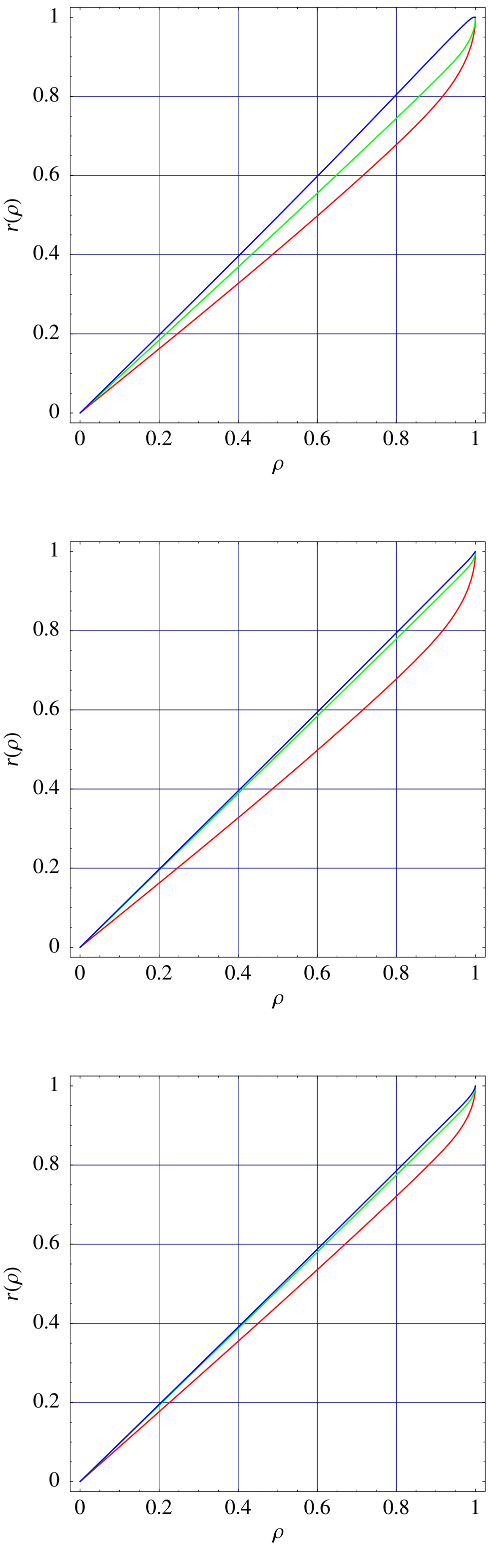

Figure 4. This figure shows the correlator output $r(\rho)$ vs. ρ for selected hybrid cases (m-level $\times$ n-level). **(Top)** 3×3-, 5×5-, and 3×5-level (red, green, blue, resp.); **(Middle)** 3×3-, 9×9-, and 3×9-level (red, green, blue, resp.); and **(Bottom)** 4×4-, 8×8-, and 4×8-level (red, green, blue, resp.).