To: file   acf-to-spectrum.wpd
From: Jon Hagen
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Subject: Transforming correlation functions to spectra

I. Recovering a Power Spectrum from a Sampled Autocorrelation Function
The power spectrum, by definition, is the Fourier Transform of the autocorrelation function
(ACF). The inverse transformation is written as

$$R(\tau) \quad ' \quad \int_{-4}^{4} S(\omega) e^{j\omega\tau} d\omega. \qquad\qquad 101)$$

For a band-limited spectrum, the integral becomes

$$R(\tau) \quad ' \quad \int_{-\omega_B}^{\omega_B} S(\omega) e^{j\omega\tau} d\omega. \qquad\qquad 102)$$

In Equation 102, $S(\omega)$ can be replaced by $S_p(\omega)$, a periodic function created by repeating $S(\omega)$,
with a period of $2\omega_P$, i.e.

$$R(\tau) \quad ' \quad \int_{-\omega_B}^{\omega_B} S_P(\omega) e^{j\omega\tau} d\omega, \qquad\qquad 103)$$

since $S(\omega)$ and $S_p(\omega)$ are identical in the range of integration. Now the function $S_p(\omega)$, because it
is periodic, can be represented by the Fourier Series

$$S_P(\omega) \quad ' \quad \sum_{n' -4}^{4} a_n \, e^{-j\frac{2\pi n\omega}{2\omega_B}} \qquad\qquad 104)$$

where the Fourier coefficient a_n is found, in the usual way, to be

$$a_n \quad ' \quad \frac{1}{2\omega_B} \int_{-\omega_B}^{\omega_B} S_p(\omega) \, e^{j\frac{2\pi n\omega}{2\omega_B}} \, d\omega \qquad\qquad 105)$$

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Comparing Equation 105 with Equation 103, we see that

$$a_n \ ' \ \frac{1}{2\omega_B} \ R\left(\tau' \ \frac{2\pi n}{2\omega_B}\right), \qquad 106)$$

showing that discrete samples of $R(\tau)$ are sufficient to calculate the periodic function $S_P(\omega)$, and hence $S(\omega)$, for any value of ω. Substituting Equation 106 into Equation 104, we have

$$S_P(\omega) \ ' \ \frac{1}{2\omega_B} \ \sum_{n' \&4}^{4} \ R\left(\tau' \ \frac{2\pi n}{2\omega_B}\right) e^{\&j\frac{2\pi n \omega}{2\omega_B}} \qquad 107)$$

Of course we need only calculate $S_P(\omega)$ for values of ω such that $-\omega_B < \omega < \omega_B$, i.e. the original band-limited spectrum. Note also that the sampled ACF points extend only to some τ_{max}, so the spectrum we calculate will have been effectively convolved with a smoothing function.

Suppose we have N points on the ACF, corresponding to N positive values of τ, including $\tau = 0$. Since $R(\tau) = R(-\tau)$, we can assemble a two-sided ACF with a total of 2N-1 points. Using Equation 107 to find the spectrum, we have

$$S_P(\omega) \ ' \ \frac{1}{2\omega_B} \ \sum_{n' \&(N\&1)}^{N\&1} \ R\left(\tau' \ \frac{2\pi n}{2\omega_B}\right) e^{\&j\frac{2\pi n \omega}{2\omega_B}} \qquad 108)$$

Again, since $R(\tau) = R(-\tau)$, we can arrange the sum to run over just the positive values of n:

$$S_P(\omega) \ ' \ \frac{1}{2\omega_B} \left[\&R(0) \ \% \ 2 \ Re \sum_{n' \ 0}^{N\&1} R\left(\tau' \ \frac{2\pi n}{2\omega_B}\right) e^{\&j\frac{2\pi n \omega}{2\omega_B}} \right] \qquad 109)$$

Normally we want to evaluate $S_P(\omega)$ at N points, from $\omega = 0$ to $\omega = (N-1)\omega_B/N$. Letting $\omega_m = m \ \omega_B/N$, we have

$$S_P(\omega_m) \ ' \ \frac{1}{2\omega_B} \left[\&R_0 \ \% \ 2 \ Re \sum_{n' \ 0}^{N\&1} R_n \ e^{\&j\frac{2\pi n m}{2N}} \right] \qquad 110)$$

If we want

to calculate the sum in Equation 110 via an FFT, we can zero-extend the ACF by defining $R_n = 0$ for $N-1 < n < 2N$. Equation 110 then becomes

$$S_P(\omega_m) \; ' \; \frac{1}{2\omega_B}\left[-R_0 \; + \; 2 \; Re \sum_{n'=0}^{2N-1} R_n \; e^{-j\frac{2\pi n m}{2N}} \right].$$ 111)

The sum within the brackets now has the desired form. The transform will produce 2N points, the first N of which are the desired spectral points. Note that the N+1 term is also valid, providing a spectral point at ω_B. The last N-1 points can be ignored.

II. Recovering a Cross Spectrum from Sampled Autocorrelation Functions

Let us denote the two signals as $V_X(t)$ and $V_Y(t)$. These could correspond to x-polarization and y-polarization. Normally we produce the N positive tau (including zero) cross correlation values in one correlator unit and N negative tau (including zero) cross correlation values in another correlation unit, i.e.

$$R^+_{XY}(n) = \text{estimate of} < V_X(t)V_Y(t + 2\pi n/(2\,\omega_B)) >$$

$$R^-_{XY}(n) = \text{estimate of} < V_X(t)V_Y(t - 2\pi n/(2\,\omega_B)) >$$

The zero lag is included in both sets. Note that the two zero lags are identical, as they are formed from sums of identical products. Together, these two sets of lagged products give us the two-sided function, $R_{XY}(n)$ for $-N < n < N$.

Let us denote the cross spectrum as $U(\omega) + jV(\omega)$. The cross spectrum is, by definition, the Fourier Transform of the cross-correlation function. Writing the inverse transform, we have

$$R_{XY}(\tau) \; ' \; \int_{-\infty}^{\infty}\left[U(\omega) + jV(\omega)\right]e^{j\omega t}d\omega$$ 201)

where U and V are the real and imaginary parts of the cross spectrum. When $V_X(t)$ and $V_Y(t)$ are band limited, their cross spectrum will also be band limited and we can write

$$R_{XY}(\tau) \; ' \; \int_{-\omega_B}^{\omega_B}\left[U(\omega) + jV(\omega)\right]e^{j\omega t}d\omega$$ 202)

Using the previous arguments, this band limited function can be represented as a Fourier series:

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$$U(\omega) \% j V(\omega) \ ' \ \sum_{n' \&4}^{4} a_n \ e^{\&j\frac{2\pi n\omega}{2\omega_B}} \qquad 204)$$

where the coefficients, a_n, are given by

$$a_n \ ' \ \frac{1}{2\omega_B} \int_{\&\omega_B}^{\omega_B} [U(\omega) \% j V(\omega)] \ e^{j\frac{2\pi n\omega}{2\omega_B}} \ d\omega \qquad 205)$$

Comparing Equation 205 with Equation 202, we see that

$$a_n \ ' \ \frac{1}{2\omega_B} R_{XY}\left(\tau' \ \frac{2\pi n}{2\omega_B}\right), \qquad 206)$$

showing that discrete samples of $R^{XY}(\tau)$ are sufficient to calculate the function $U(\omega) + jV(\omega)$ for any value of ω. Substituting Equation 206 into Equation 204, we have

$$U(\omega) \% j V(\omega) \ ' \ \frac{1}{2\omega_B} \sum_{n' \&4}^{4} R_{XY}\left(\tau' \ \frac{2\pi n}{2\omega_B}\right) e^{\&j\frac{2\pi n\omega}{2\omega_B}} \qquad 207)$$

Normally we want to evaluate the cross spectrum for N points, starting with $\omega=0$ and separated by a spacing ω_B/N. The mth cross spectral point is given by

$$U(m) \% j V(m) \ ' \ \frac{1}{2\omega_B} \sum_{n' \&(N\&1)}^{N\&1} R_{XY}(n) \ e^{\&j\frac{2\pi nm}{2N}} \qquad 208)$$

$$' \ \frac{1}{2\omega_B} \left[\sum_{n' 0}^{N\&1} R^{\%}{}_{XY}(n) \ e^{\&j\frac{2\pi nm}{2N}} \% \sum_{n' 0}^{N\&1} R^{\&}{}_{XY}(n) \ e^{j\frac{2\pi nm}{2N}} \&R_{XY}(0) \right], \qquad 209)$$

where $R_{XY}(0)$ is either $R^+{}_{XY}(0)$ or $R^-{}_{XY}(0)$, since they are identical. To put this into the standard

form for evaluation via the FFT algorithm, we can add zeros to R^+_{XY} and R^-_{XY} to extend both to length 2N, i.e. $R^+_{XY}(n) = R^-_{XY}(n) = 0$ for $N-1 < n < 2N$. The cross spectrum becomes

$$U(m) \% jV(m) \,' \, \frac{1}{2\omega_B} \left[\sum_{n'0}^{2N\&1} R^{\%}_{XY}(n) \; e^{\&j\frac{2\pi n m}{2N}} \; \% \; \sum_{n'0}^{2N\&1} R^{\&}_{XY}(n) \; e^{j\frac{2\pi n m}{2N}} \; \& R_{XY}(0) \right] \quad 210)$$

As before, the first N+1 terms provide cross spectrum values for equally spaced values of ω from zero to ω_B.